

A TWISTED $\bar{\partial}_f$ -NEUMANN PROBLEM AND TOEPLITZ n -TUPLES FROM SINGULARITY THEORY

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ABSTRACT. A twisted $\bar{\partial}_f$ -Neumann problem associated to a singularity (\mathcal{O}_n, f) is established. By constructing the connection to the Koszul complex for toeplitz n -tuples (f_1, \dots, f_n) on Bergman spaces $B^0(D)$, we can solve this $\bar{\partial}_f$ -Neumann problem. Moreover, the cohomology of the L^2 holomorphic Koszul complex $(B^*(D), \partial f \wedge)$ can be computed explicitly.

1. INTRODUCTION

Let D be a bounded pseudoconvex domain in \mathbb{C}^n with C^∞ smooth boundary ∂D and f a holomorphic function on \bar{D} with only isolated critical points in D and no critical points on ∂D . Under such assumption, we get two objects in the framework of analysis.

The first object is the toeplitz n -tuples with symbols (f_1, f_2, \dots, f_n) defined on the Bergman space on D , where the f_i 's are partial derivatives of f . One can study the L^2 holomorphic complex $(B^*(D), \partial f \wedge)$ given by

$$0 \rightarrow B^0(D) \xrightarrow{\partial f \wedge} B^1(D) \xrightarrow{\partial f \wedge} \dots \xrightarrow{\partial f \wedge} B^n(D) \rightarrow 0.$$

Note that if without L^2 condition, this complex is an algebraic Koszul complex. If assuming (f_1, \dots, f_n) is regular, then the homology of the algebraic Koszul complex will only be nontrivial on the top term and is isomorphic to the Jacobian ring of f on D . In the assumption of L^2 integrability, lack of noetherian ring structure make things complicated. This complex is an important example in Taylor's multivariable spectral theory (ref.[Ta]) and has been studied a lot. The spectral picture, spectral mapping theorem and the index theory were all developed (ref.[EP]). The index of this complex is computed to be the dimension of $\text{Jac}(f)$ on D (ref.[EP], Chapter 10). The fact that the cohomology is concentrated at the n^{th} degree should be known (we were informed by M. Putinar [Pu] that this can be proved via the spectral

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localization technique), but the direct proof seems not so easy. In this paper, we will reprove this result via the study of $\bar{\partial}_f$ operator.

On the other hand, we can define the twisted Cauchy-Riemann operator $\bar{\partial}_f := \bar{\partial} + \partial f \wedge$ on D , which only preserving the real grading of the differential forms, not the Hodge grading. This operator was used by physicists to study the topological field theory of Landau-Ginzburg model from the B side (ref. [Ce, CV]). In recent years, LG model has been found to be a very important part of 2-d topological field theory, mirror symmetry and categorification theory of open strings (ref. [FJR, CR, FJ], [GMW], [KKS]). Inspired by the physicists' work, the second author proposed an approach ([Fa]) to study the singularity theory of f by constructing the Hodge theory for the operator $\bar{\partial}_f$ and the twisted Laplacian $\Delta_f = \bar{\partial}_f \bar{\partial}_f^* + \bar{\partial}_f^* \bar{\partial}_f$. The aim is to construct the Saito's Frobenius manifold structure (ref. [ST]) for singularities and eventually treat the quantization problem of LG model from the B side. Recently, a different method via the theory of polyvector fields was built by Li-Li-Saito [LLS] for studying the singularity and the related primitive forms, which however did not touch the Hodge structure. The paper [Fa] can only treat the marginal deformation of a general singularity, but not the universal deformation of a singularity. Hence to recover Saito's Frobenius manifold structure from the analytical method, we must study some boundary value problem of $\bar{\partial}_f$ operator.

In this paper, we will study the $\bar{\partial}_f$ -Neumann problem on D . This problem is related to the L^2 complex $(L^2(D), \bar{\partial}_f)$, whose cohomology group is denoted by $H_{((2), \bar{\partial}_f)}^*$. As the first result, we can solve the $\bar{\partial}_f$ -Neumann problem by proving the strong Hodge decomposition theorem as below.

Theorem 1.1. *Let D be a bounded strongly pseudoconvex domain in \mathbb{C}^n with C^∞ smooth boundary ∂D and f a holomorphic function on \bar{D} with only isolated critical points in D and no critical points on ∂D . Then we have the decomposition*

$$H^*(D) = \mathcal{H}^* \oplus \text{im } \bar{\partial}_f \oplus \text{im } \bar{\partial}_f^*. \quad (1.1)$$

and then the isomorphism

$$H_{((2), \bar{\partial}_f)}^* \cong \mathcal{H}^*. \quad (1.2)$$

Furthermore, all the spaces \mathcal{H}^* are of finite dimensional.

Theorem 1.1 is a direct conclusion of Theorem 3.7 and Corollary 3.8. To prove this theorem, we first show that Δ_f with $\bar{\partial}_f$ -Neumann boundary condition is a self-adjoint operator, hence there exists a weak

Hodge decomposition. Usually, to prove the strong decomposition, we need a global a priori estimate for the Green operator which can naturally deduce the compactness by Rellich theorem. However, things are different in our $\bar{\partial}_f$ -Neumann problem. The Δ_f operator always mix $(k, 0)$ -forms with other types of forms, thus the a priori estimate becomes complicated because there is no global estimate to control the Sobolev norms of holomorphic k form in $\bar{\partial}$ -Neumann problem. However, we have an indirect way to get around this problem. We can construct an isomorphism between the L^2 complex $(L^2(D), \bar{\partial}_f)$ and the L^2 holomorphic complex $(B^*(D), \partial f \wedge)$. By Taylor's joint spectral theory, the cohomology of the later complex can be proved to be of finite dimension. Using the finite dimensionality and a theorem from functional analysis, we can prove the range of $\bar{\partial}_f$ and $\bar{\partial}_f^*$ are all closed. So this proves the strong Hodge decomposition, meanwhile we can prove that the spectrum of Δ_f has a gap at 0.

Conversely, by studying the complex $(L^2(D), \bar{\partial}_f)$ in C^∞ category, we can calculate the cohomology of $(B^*(D), \partial f \wedge)$ as mentioned above. The second main result is as follows.

Theorem 1.2. *If D is a bounded strongly pseudoconvex domain in \mathbb{C}^n with C^∞ smooth boundary ∂D and f is a holomorphic function on \bar{D} with isolated critical points in D and no critical points on ∂D , then the dimension of the Koszul cohomology on Bergman spaces is concentrated at the n^{th} degree and equal to the number of critical points, with multiplicities accounted, of f in D .*

Corollary 1.3. *Under the assumption of Theorem 1.2, The cohomology groups*

$$H_{\bar{\partial}_f}^*(D), H_{\bar{\partial}_f}^*(\bar{D}), H_{(c, \bar{\partial}_f)}^*, H_{((2), \bar{\partial}_f)}^*, H_{\bar{\partial}_f}^*(\mathbb{C}), H_{(0, \bar{\partial}_f)}^*$$

are all isomorphic to the space \mathcal{H}^ .*

Remark 1.4. For arbitrary n -tuples (f_1, f_2, \dots, f_n) satisfying the condition that they have only finite common zeros and have no common zeros on ∂D , the proof in our article can be applied to the operator $\bar{\partial} + (f_1 dz_1 + \dots + f_n dz_n) \wedge$ and all our results still holds. In fact, throughout our article, we will not use the fact that the f_i 's are the partial derivatives of a single function.

Notation 1.5. We use the super-bracket

$$[A, B] := AB - (-1)^{\deg(A) \deg(B)} BA$$

in this paper.

2. $\bar{\partial}_f$ -NEUMANN PROBLEM

Let $h = \sum_i \frac{1}{2} dz^i \otimes dz^{\bar{i}}$ be the standard hermitian metric of \mathbb{C}^n in the coordinate system $\{z_i, i = 1, \dots, n\}$. Let D be a bounded strongly pseudoconvex domain in \mathbb{C}^n with smooth boundary and f a holomorphic function on \bar{D} .

The study of pseudoconvexity is one of the central topic in the theory of functions of several complex variables. D is called pseudoconvex if it can be exhausted by a continuous plurisubharmonic function. Every (geometrically) convex domain in \mathbb{C}^n is pseudoconvex. If the boundary ∂D is C^2 , then this is equivalent to the Levi pseudoconvexity we will explain below.

Let r be a C^2 function defined in a neighborhood of $p \in \partial D$ satisfying $r|_{\partial D} = 0$ and $\|dr\| = 1$ on ∂D . Then we can define a Levi form L_p along the $n - 1$ dimensional subspace $\{\xi \in T_p^{\mathbb{C}}\mathbb{C}^n \mid \sum_{j=1}^n \frac{\partial r}{\partial z^j} \xi^j = 0\}$ by

$$L_p(\xi, \eta) = \sum_{i,j} \frac{\partial^2 r}{\partial z^i \partial \bar{z}^j} \xi^i \bar{\eta}^j. \quad (2.1)$$

If the Levi form L_p is semi-positive at all points $p \in \partial D$, then D is said to be Levi pseudoconvex. If L_p is positive at all points of ∂D , then D is said to be strongly pseudoconvex. The balls in \mathbb{C}^n are strongly pseudoconvex.

Denote by $\mathcal{A}^p(D)$ (or $\mathcal{A}^{p,q}(D)$) the space of smooth p forms (or (p, q) -forms) on D and $\mathcal{A}(D) = \bigoplus_p \mathcal{A}^p(D)$. Let $\mathcal{A}^p(\bar{D})$ be the subspace of $\mathcal{A}^p(D)$ whose elements can be extended smoothly to a small neighborhood of \bar{D} and $\mathcal{A}(\bar{D}) = \bigoplus_p \mathcal{A}^p(\bar{D})$. $\mathcal{A}_c^p(D)$ is a subspace of $\mathcal{A}^p(\bar{D})$ whose elements have compact support disjoint from ∂D . Similarly, we have the definitions of $\mathcal{A}^{p,q}(\bar{D})$ and $\mathcal{A}_c^{p,q}(\bar{D})$.

For any form $\varphi \in \mathcal{A}^{p,q}(\bar{D})$, we have the expression

$$\varphi = \sum'_{I,J} \varphi_{I,J} dz^I \wedge d\bar{z}^J,$$

where \sum' means summation over strictly increasing multi-indices and $\varphi_{I,J}$'s are antisymmetric for arbitrary I and J .

For any (p, q) -forms $\varphi = \sum'_{I,J} \varphi_{I,J} dz^I \wedge d\bar{z}^J$ and $\psi = \sum'_{I,J} \psi_{I,J} dz^I \wedge d\bar{z}^J$, we can define the L^2 inner product:

$$\langle \varphi, \psi \rangle = \sum'_{I,J} \langle \varphi_{I,J}, \psi_{I,J} \rangle = \sum'_{I,J} \int_D \varphi_{I,J} \overline{\psi_{I,J}} dV$$

where dV denote the volume element on D defined by h .

Let $\|\cdot\|$ be the corresponding L^2 -norm and $L^2_{(p,q)}(D)$ be the L^2 -completion space of $\mathcal{A}^{p,q}(\bar{D})$. Define $L^2_k(D) = \oplus_{p+q=k} L^2_{(p,q)}(D)$ and $L^2(D) = \oplus_k L^2_k(D)$. Furthermore, the Sobolev s -norms $\|\cdot\|_s$ and the corresponding Sobolev spaces $W^s_{(p,q)}(D)$, $W^s_k(D)$, $W^s(D)$ can be defined. For example, for non-negative integer s , elements of $W^s(D)$ has derivatives in $L^2(D)$ up to s order and $\|\varphi\|_s$ is the sum of L^2 norms of derivatives of φ up to s order. In particular, we have $W^0(D) = L^2(D)$.

Now any differential operator T defined on $\mathcal{A}(\bar{D})$ can be extended to a unbounded closed operator in $L^2(D)$ by means of generalized derivatives. Remember that if T is a closed operator defined on $\text{Dom}(T) \subset L^2(D)$ if and only if the following holds: if $\varphi_i \in \mathcal{A}^p(\bar{D}) \cap L^2(D)$ and $T(\varphi_i) \in L^2(D)$ are function sequences such that $\varphi_i \rightarrow \varphi$ and $T(\varphi_i) \rightarrow \psi \in L^2(D)$, then φ is in $\text{Dom}(T)$ and $T(\varphi) = \psi$.

Now the Cauchy-Riemann operator $\bar{\partial}$ and the twisted operator operator $\bar{\partial}_f = \bar{\partial} + \partial f \wedge : \mathcal{A}^k(\bar{D}) \rightarrow \mathcal{A}^k(\bar{D})$ can be extended to closed operators in $L^2(D)$ such that

$$\begin{aligned} \text{Dom}(\bar{\partial}) &= \{\varphi \in L^2(D) | \bar{\partial}\varphi \in L^2(D)\} \\ \text{Dom}(\bar{\partial}_f) &= \{\varphi \in L^2(D) | \bar{\partial}_f\varphi \in L^2(D)\}. \end{aligned}$$

Since f is bounded on D , the multiplication operator $\partial f \wedge$ has the domain $\text{Dom}(\partial f \wedge) = L^2(D)$ and actually we have

$$\text{Dom}(\bar{\partial}_f) = \text{Dom}(\bar{\partial}).$$

Now we consider the adjoint of $\bar{\partial}_f$ under the L^2 norm. By definition the Hilbert space adjoint $\bar{\partial}_f^*$ of $\bar{\partial}_f$ is defined on the domain $\text{Dom}(\bar{\partial}_f^*)$ consisting of all $\varphi \in L^2_k(D)$ such that $|\langle \varphi, \bar{\partial}_f(\psi) \rangle| \leq c\|\psi\|$ for some positive constant c and for all $\psi \in \text{Dom}(\bar{\partial}_f)$. As $\partial f \wedge$ is bounded, the above inequality is equivalent to $|\langle \varphi, \bar{\partial}(\psi) \rangle| \leq c'\|\psi\|$ for some positive constant c' . This means $\text{Dom}(\bar{\partial}_f^*) = \text{Dom}(\bar{\partial}^*)$ and they have the same Neumann boundary conditions.

For $\varphi, \psi \in \mathcal{A}(\bar{D})$, we have the integration by parts:

$$\langle \bar{\partial}_f \varphi, \psi \rangle = \langle \varphi, \vartheta_f \psi \rangle + \int_{\partial D} \langle \sigma(\bar{\partial}, dr) \varphi, \psi \rangle \quad (2.2)$$

$$\langle \vartheta_f \varphi, \psi \rangle = \langle \varphi, \bar{\partial}_f \psi \rangle + \int_{\partial D} \langle \sigma(\vartheta, dr) \varphi, \psi \rangle. \quad (2.3)$$

Here

$$\vartheta_f = \vartheta + \bar{f}_j \iota_{\partial_j},$$

where ϑ represents the formal adjoint of $\bar{\partial}$ and ι_{∂_j} is the contraction operator with the vector $\partial_j = \frac{\partial}{\partial \bar{z}^j}$, and

$$\sigma(\bar{\partial}, dr) = \bar{\partial} r \wedge = \frac{\partial r}{\partial \bar{z}^j} d\bar{z}^j \wedge, \quad \sigma(\vartheta, dr) = -\frac{\partial r}{\partial z_j} \iota_{\partial_j}.$$

Hence we have

$$\text{Dom}(\bar{\partial}_f^*) \cap \mathcal{A}(\bar{D}) = \{\varphi \in \mathcal{A}(\bar{D}) | \sigma(\vartheta, dr)\varphi = 0 \text{ on } \partial D\}$$

Denote by $\mathcal{D}^{p,q} = \{\varphi \in \mathcal{A}(\bar{D}) | \sigma(\vartheta, dr)\varphi = 0 \text{ on } \partial D\}$ and $\mathcal{D}^k = \bigoplus_{p+q=k} \mathcal{D}^{p,q}$.

Definition 2.1. Let $\Delta_f = [\bar{\partial}_f, \bar{\partial}_f^*] = \bar{\partial}_f \bar{\partial}_f^* + \bar{\partial}_f^* \bar{\partial}_f$ be the operator from $L^2(D)$ to $L^2(D)$ with domain $\text{Dom}(\Delta_f) = \{\varphi \in L^2(D) | \varphi \in \text{Dom}(\bar{\partial}_f) \cap \text{Dom}(\bar{\partial}_f^*); \bar{\partial}_f(\varphi) \in \text{Dom}(\bar{\partial}_f^*) \text{ and } \bar{\partial}_f^*(\varphi) \in \text{Dom}(\bar{\partial}_f)\}$.

Proposition 2.2. Δ_f is a linear, densely defined, closed self-adjoint operator.

Proof. The proof is the same as the proof of Proposition 1.3.8 in [FK]. \square

Remark 2.3. We can consider the formal Laplacian $\hat{\Delta}_f = \bar{\partial}_f \vartheta_f + \vartheta_f \bar{\partial}_f + I$ defined on $\mathcal{D}^{p,q}$. This operator has a unique Friedrichs self-adjoint extension related to the quadratic form $Q(\varphi, \phi) = (\bar{\partial}_f \varphi, \bar{\partial}_f \psi) + (\vartheta_f \varphi, \vartheta_f \psi) + (\varphi, \psi)$. This extended self-adjoint operator is just $\Delta_f + I$ and the equivalence relation is clear by the standard abstract theorem in functional analysis.

The self-adjointness of Δ_f is due to the $\bar{\partial}$ -Neumann boundary condition which is characterized by

$$\begin{aligned} \text{Dom}(\Delta_f) \cap \mathcal{A}(\bar{D}) &= \{\varphi \in \mathcal{A}(\bar{D}) | \sigma(\vartheta, dr)\varphi = 0 \text{ and} \\ &\quad \sigma(\vartheta, dr)\bar{\partial}\varphi = 0 \text{ on } \partial D\}. \end{aligned} \quad (2.4)$$

Similar to the $\bar{\partial}$ -Neumann problem, here we want to solve the equation $\Delta_f \varphi = \eta \in L^2(D)$ under the $\bar{\partial}$ -Neumann boundary condition. We call this as $\bar{\partial}_f$ -Neumann problem.

Since Δ_f is self-adjoint and $\text{Im}(\bar{\partial}_f) \perp \text{Im}(\bar{\partial}_f^*)$, we get a weak Hodge decomposition

$$L_k^2(D) = \mathcal{H}^k \oplus \overline{\text{Im}(\Delta_f)} = \mathcal{H}^k \oplus \overline{\text{Im}(\bar{\partial}_f)} \oplus \overline{\text{Im}(\bar{\partial}_f^*)} \quad (2.5)$$

where \mathcal{H}^k denote the kernel of Δ_f .

To solve the $\bar{\partial}_f$ -Neumann problem, we need to prove that all the range in the above decomposition are closed. The $\bar{\partial}_f$ -Neumann problem will display different nature compared to the $\bar{\partial}$ -Neumann problem, in which f will play dominant role. This will be shown in next section.

3. $\bar{\partial}_f$ -COMPLEXES, FINITE DIMENSIONALITY AND SPECTRAL GAP

In this section, we first discuss various $\bar{\partial}_f$ complexes defined on a bounded pseudoconvex domain. Then we will show that the L^2 $\bar{\partial}_f$ -complex has finite dimensional cohomology groups and there exists a spectral gap between 0 and other spectra of Δ_f . In the $\bar{\partial}$ -Neumann problem, there is no estimate for L^2 integrable holomorphic $(p, 0)$ -forms, which is in the kernel of $\Delta_{\bar{\partial}}$, near the boundary. For this reason, we solve the $\bar{\partial}_f$ -Neumann problem in an indirect way. We avoid to estimate directly the behavior of the operator Δ_f , which twist the $(p, 0)$ -forms and other types of forms, instead, we will use a classical result in multivariable spectra theory about the complex $(B^*(D), \partial f \wedge)$ and some results in the theory of unbounded linear operators.

3.1. $\bar{\partial}_f$ -complexes.

There are various $\bar{\partial}_f$ -complexes which are defined by smoothness or boundary value conditions. At first, we have the L^2 $\bar{\partial}_f$ -complex

$$L^2(D) : L^2_0(D) \xrightarrow{\bar{\partial}_f} L^2_1(D) \xrightarrow{\bar{\partial}_f} L^2_2(D) \xrightarrow{\bar{\partial}_f} \dots \quad (3.1)$$

corresponding to L^2 integrable p -forms. The cohomology group is defined as

$$H^k_{((2), \bar{\partial}_f)} = \frac{\{\varphi \in \text{Dom}(\bar{\partial}_f) \mid \bar{\partial}_f \varphi = 0\}}{\bar{\partial}_f(\text{Dom}(\bar{\partial}_f))}$$

In addition, there are $\bar{\partial}_f$ -complexes $\mathcal{A}^*(D), \mathcal{A}^*(\bar{D}), \mathcal{A}^*_c(D)$, which correspond to smooth p -form on D , on \bar{D} , and having compact support in D respectively. We denote by $H^k_{\bar{\partial}_f}(\bar{D}), H^k_{\bar{\partial}_f}(D), H^k_{(c, \bar{\partial}_f)}(D)$ the corresponding cohomology groups.

Let $\mathcal{C}^{p,q} = \{\varphi \in \mathcal{A}(\bar{D}) \mid \sigma(\bar{\partial}, dr)\varphi = 0 \text{ on } \partial D\}$ and $\mathcal{C}^k = \bigoplus_{p+q=k} \mathcal{C}^{p,q}$. We can take ϑ_f as a closed operator in $H(D)$ at first and then consider the ϑ_f -Neumann problem, and in this case, we have $\mathcal{C}^k = \mathcal{A}^k(\bar{D}) \cap \text{Dom}(\vartheta_f^*)$.

Lemma 3.1. $\bar{\partial}_f \mathcal{C}^k \subset \mathcal{C}^{k+1}$.

Proof. If $\psi \in \mathcal{C}^k$, then it can be written as $\psi = \bar{\partial} r \wedge \alpha + r \beta$ for $\alpha \in \mathcal{A}^{k-1}(\bar{D}), \beta \in \mathcal{A}^k(\bar{D})$. Then $\bar{\partial}_f \psi = \bar{\partial} r \wedge (\beta - \bar{\partial}_f \alpha) + r \bar{\partial}_f \beta$ which is in \mathcal{C}^k . \square

This lemma shows that $(\mathcal{C}^*, \bar{\partial}_f)$ forms a complex and has cohomology $H^k_{\bar{\partial}_f}(\mathcal{C})$.

As in [FK], we also have the Dirichlet or zero-boundary value cohomology

$$H_{(0,\bar{\partial}_f)}^k = \frac{\{\psi \in \mathcal{A}^k(\bar{D}) \mid \bar{\partial}_f \psi = 0, \psi|_{\partial D} = 0\}}{\bar{\partial}_f \{\psi \in \mathcal{A}^{k-1}(\bar{D}) \mid \psi|_{\partial D} = 0, \bar{\partial}_f \psi|_{\partial D} = 0\}}. \quad (3.2)$$

Proposition 3.2. *There exists isomorphism $i : H_{(0,\bar{\partial}_f)}^k \cong H_{\bar{\partial}_f}^k(\mathbb{C})$.*

Proof. Suppose that $\phi \in \mathcal{A}^k(\bar{D})$, $\phi|_{\partial D} = 0$, and $\phi = \bar{\partial}_f \psi$ with $\psi \in \mathcal{C}^{k-1}$. Then ψ has the form $\psi = \bar{\partial} r \wedge \alpha + r\beta$. This can be rewritten as

$$\psi = \bar{\partial}_f(r\alpha) + r(-\bar{\partial}_f \alpha + \beta).$$

Let $\psi_0 = r(-\bar{\partial}_f \alpha + \beta)$. This gives $\phi = \bar{\partial}_f \psi_0$, which shows that i is a well-defined injective map. To prove the surjectivity, suppose $\phi \in \mathcal{C}^k$ and $\bar{\partial}_f \phi = 0$. Then ϕ also has the expression $\phi = \bar{\partial}_f(r\alpha) + r(-\bar{\partial}_f \alpha + \beta)$. Hence ϕ is cohomologous to $r(-\bar{\partial}_f \alpha + \beta)$, which vanishes on ∂D . \square

We will discuss other relations between these cohomologies in the following sections. Above all, we want to discuss the relation between the L^2 complex $(L^2(D), \bar{\partial}_f)$ and the L^2 holomorphic Koszul complex $(B^*(D), \partial f \wedge)$.

3.2. Koszul complex, finite dimensionality and spectral gap.

Let $B^k(D)$ be the L^2 integrable holomorphic k -form on D , i.e., $B^0(D)$ is the Bergman space on D and $B^k(D)$ can be viewed as direct products of $B^0(D)$. The complex $(B^*(D), \partial f \wedge)$ is defined as

$$0 \rightarrow B^0(D) \xrightarrow{\partial f \wedge} B^1(D) \xrightarrow{\partial f \wedge} \dots \xrightarrow{\partial f \wedge} B^n(D) \rightarrow 0,$$

whose cohomology are denoted by $H_{\partial f \wedge}^*(D)$.

In 1970, J. L. Taylor [Ta] developed a multivariable (joint) spectral theory. Given a Hilbert space X and a commuting n -tuples of bounded linear operators $T = (T_1, \dots, T_n)$ on X , the joint spectra $\sigma(T, X)$ is the set of all $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that $K^*(T - \lambda, B(D))$ is not acyclic. The essential joint spectra $\sigma_e(T, X)$ is the set of all λ such that the cohomology of $K^*(T - \lambda, B(D))$ is not finite dimensional. The finite complex $K^*(T - \lambda, B(D))$ consists of the spaces

$$K^p(T - \lambda, X) = X \otimes_{\mathbb{C}} \Lambda^p(\mathbb{C}^n) \quad (0 \leq p \leq n)$$

and the coboundary operators

$$d^p : K^p(T - \lambda, X) \rightarrow K^{p+1}(T - \lambda, X), \quad d^p(\varphi) = \tau \wedge \varphi$$

where $\tau = (T_1 - \lambda_1) \otimes e_1 + (T_2 - \lambda_2) \otimes e_2 + \dots + (T_n - \lambda_n) \otimes e_n$ and (e_1, e_2, \dots, e_n) is the canonical basis of \mathbb{C}^n .

The L^2 $\partial f \wedge$ -complex $(B^*(D), \partial f \wedge)$ can be viewed as a model for Taylor's joint spectral theory. The Bergman space is a Hilbert space and the toeplitz operators defined by multiplication by $f_i = \frac{\partial f}{\partial z_i}$, $1 \leq i \leq n$, is a commuting n -tuples of bounded linear operators. The dz_i 's can be viewed as a basis of \mathbb{C}^n . Thus the associated Koszul complex is exactly $(B^*(D), \partial f \wedge)$.

Under our assumption, D is bounded and pseudoconvex. By Theorem 8.1.1 and corollary 8.1.2 of [EP], we have

$$\sigma(z_1, \dots, z_n, B^0(D)) = \bar{D}$$

and

$$\sigma_e(z_1, \dots, z_n, B^0(D)) \subset \partial D.$$

Furthermore by Theorem 8.2.1 and Proposition 8.2.5 of [EP], we have

$$\sigma(f_1, \dots, f_n, B^0(D)) = \overline{(f_1, \dots, f_n)(D)}$$

and

$$\begin{aligned} \sigma_e(f_1, \dots, f_n, B^0(D)) &= (f_1, \dots, f_n)(\sigma_e(z_1, \dots, z_n, B^0(D))) \\ &\subset (f_1, \dots, f_n)(\partial D). \end{aligned}$$

Hence we have the simple conclusion:

Proposition 3.3. *Assume that f is holomorphic on \bar{D} and has no critical points on ∂D , then*

$$0 \notin \sigma_e(f_1, \dots, f_n, B^0(D)),$$

which says that the complex $(B^(D), \partial f \wedge)$ has at most finite dimensional cohomology group.*

Now we turn to the discussion of the L^2 complex $(H^*(D), \bar{\partial}_f)$. The key theorem in this section is as follows.

Theorem 3.4. *There exists a quasisomorphism between the L^2 complex $(L^2(D), \bar{\partial}_f)$ and the complex $(B^*(D), \partial f \wedge)$. Moreover, their p^{th} cohomology group vanishes for $n < p \leq 2n$.*

We are working in L^2 integrable category, so we must be careful to control the norms. Before proving Theorem 3.4, we need the L^2 existence theorem for $\bar{\partial}$ -Neumann problem *.

*The assumption $n \geq 2$ doesn't matter in our references. This is because when $n = 1$, the $\bar{\partial}$ -Neumann condition is exactly the 0-value Dirichlet condition for $(p, 1)$ -forms and all the existence and regularity theorems clearly hold by standard elliptic estimate.

Theorem 3.5 ([Sh]). *Let D be a bounded pseudoconvex domain in \mathbb{C}^n with C^∞ smooth boundary. If $\bar{\partial}\varphi = 0$ for some $\varphi \in L^2_{(p,q+1)}(D)$, then there exists $\psi \in L^2_{(p,q)}(D)$ such that $\varphi = \bar{\partial}\psi$ and $\|\psi\| \leq c\|\varphi\|$. Here $0 \leq p \leq n$, $0 \leq q \leq n-1$ and c is independent of the choice of φ .*

We will also need the Banach's closed range theorem as below.

Theorem 3.6 ([Sh]). *Let $T : X \rightarrow Y$ be a closed linear operator between two Hilbert spaces and T' be the transpose of T . Then the following conditions are equivalent:*

- (1) T has closed range in Y .
- (2) T' has closed range in X .
- (3) There exists positive constant c , such that

$$\|Tx\| \geq c\|x\|, \forall x \in \text{Dom}(T) \cap \text{Ker}(T)^\perp$$

- (4) There exists positive constant c , such that

$$\|T'y\| \geq c\|y\|, \forall y \in \text{Dom}(T') \cap \text{Ker}(T')^\perp$$

Proof of Theorem 3.4. Assume $\bar{\partial}_f \varphi = 0$ for some $\varphi \in L^2_p(D)$. To avoid too heavy notation, here and below we will use $a \lesssim b$ to denote 'there exists a constant $c > 0$ such that $a \leq c \cdot b$ '.

Firstly we assume $n < p \leq 2n$. Let

$$\varphi = \varphi^{n,p-n} + \varphi^{n-1,p-n+1} + \dots + \varphi^{p-n,n}$$

Then we have $\bar{\partial}\varphi^{p-n,n} = 0$. By Theorem 3.5, there exists $\psi^{p-n,n-1}$ such that

$$\bar{\partial}\psi^{p-n,n-1} = \varphi^{p-n,n}$$

and

$$\|\psi^{p-n,n-1}\| \lesssim \|\varphi^{p-n,n}\|$$

Then

$$\begin{aligned} \partial f \wedge \varphi^{p-n,n} + \bar{\partial}\varphi^{p-n+1,n-1} &= \bar{\partial}(\varphi^{p-n+1,n-1} - \partial f \wedge \psi^{p-n,n-1}) \\ &= 0 \end{aligned}$$

Again by Theorem 3.5, there exists $\psi^{p-n+1,n-2}$ such that

$$\bar{\partial}\psi^{p-n+1,n-2} = \varphi^{p-n+1,n-1} - \partial f \wedge \psi^{p-n,n-1}$$

and

$$\begin{aligned} \|\psi^{p-n+1,n-2}\| &\lesssim \|\varphi^{p-n+1,n-1} - \partial f \wedge \psi^{p-n,n-1}\| \\ &\lesssim \|\varphi^{p-n+1,n-1}\| + \|\psi^{p-n,n-1}\| \\ &\lesssim \|\varphi^{p-n+1,n-1}\| + \|\varphi^{p-n,n}\| \end{aligned}$$

Inductively, we have $\psi^{p-n+k, n-1-k} \in L^2_{(p-n+k, n-1-k)}(D)$ such that

$$\bar{\partial}\psi^{p-n+k, n-1-k} = \varphi^{p-n+k, n-k} - \partial f \wedge \psi^{p-n+k-1, n-k}$$

and

$$\|\psi^{p-n+k, n-1-k}\| \lesssim \sum_{i=0}^k \|\varphi^{p-n+i, n-i}\|$$

Note that $\psi^{p-n-1, n} = 0$ here. Let $\psi = \sum_{k=0}^{2n-p} \psi^{p-n+k, n-1-k}$, then $\bar{\partial}_f \psi = \varphi$ and $\|\psi\| \lesssim \|\varphi\|$. This means that the L^2 $\bar{\partial}_f$ -complex is exact at p^{th} degree and thus the p^{th} cohomology vanish for $n < p \leq 2n$. Moreover, by Theorem 3.6, $\bar{\partial}_f$ has closed range at these degrees.

Then we consider the case $0 \leq p \leq n$. Let

$$\varphi = \varphi^{p,0} + \varphi^{p-1,1} + \dots + \varphi^{0,p}$$

Because $\bar{\partial}_f \varphi = 0$, we have

$$\bar{\partial}\varphi^{p-k,k} + \partial f \wedge \varphi^{p-k-1,k+1} = 0, 0 \leq k \leq p$$

Similar to the discussion above, we have $\psi^{p-1-k,k}, 0 \leq k \leq p-1$ such that

$$\bar{\partial}\psi^{p-k,k-1} = \varphi^{p-k,k} - \partial f \wedge \psi^{p-1-k,k}, 1 \leq k \leq p-1$$

and

$$\bar{\partial}(\varphi^{p,0} - \partial f \wedge \psi^{p-1,0}) = 0 \quad (3.3)$$

$$\partial f \wedge (\varphi^{p,0} - \partial f \wedge \psi^{p-1,0}) = \partial f \wedge \varphi^{p,0} = 0 \quad (3.4)$$

Let $\psi = \sum_{k=0}^{p-1} \psi^{p-1-k,k}$, then

$$\varphi = \bar{\partial}_f \psi + (\varphi^{p,0} - \partial f \wedge \psi^{p-1,0})$$

Similar to the discussion above, norm of $\varphi^{p,0} - \partial f \wedge \psi^{p-1,0}$ can be controlled by that of φ and eventually by φ . Thus by (3.3) and (3.4), $\varphi^{p,0} - \partial f \wedge \psi^{p-1,0}$ is in $B^p(D)$ and represents an element in $H^p_{\bar{\partial}_f \wedge}(D)$.

We define a map between the two complexes at the level of L^2 cohomology by:

$$u : H^p_{\bar{\partial}_f}(D) \rightarrow H^p_{\bar{\partial}_f \wedge}(D), [\varphi] \mapsto [\varphi^{p,0} - \partial f \wedge \psi^{p-1,0}]$$

If $[\varphi] = 0 \in H^p_{\bar{\partial}_f}(D)$, then we have $\varphi = \bar{\partial}_f \eta$, together with

$$\varphi = \bar{\partial}_f \psi + (\varphi^{p,0} - \partial f \wedge \psi^{p-1,0})$$

we have

$$\varphi^{p,0} - \partial f \wedge \psi^{p-1,0} = \bar{\partial}_f(\eta - \psi)$$

by counting degrees, we have

$$\varphi^{p,0} - \partial f \wedge \psi^{p-1,0} = \partial f \wedge (\eta - \psi)^{p-1,0}$$

i.e. $[\varphi^{p,0} - \partial f \wedge \psi^{p-1,0}] = 0 \in H_{\partial f \wedge}^p(D)$ and thus u is well defined.

If $\eta \in B^p(D)$ represent a cohomology class, we have $\bar{\partial}_f \eta = 0$ and $u([\eta]) = [\eta]$ and u is surjective.

If $u([\varphi]) = [\varphi^{p,0} - \partial f \wedge \psi^{p-1,0}] = 0 \in H_{\partial f \wedge}^p(D)$, then

$$\begin{aligned} \varphi &= \bar{\partial}_f \psi + (\varphi^{p,0} - \partial f \wedge \psi^{p-1,0}) \\ &= \bar{\partial}_f \psi + \partial f \wedge \theta \\ &= \bar{\partial}_f(\psi + \theta) \end{aligned}$$

i.e. $[\varphi] = 0 \in H_{\bar{\partial}_f}^p(D)$, and u is injective. Thus u is an isomorphism of cohomology groups. \square

Now by the weak Hodge decomposition

$$L_k^2(D) = \mathcal{H}^k \oplus \overline{Im(\bar{\partial}_f)} \oplus \overline{Im(\bar{\partial}_f^*)}$$

we have

$$H_{((2), \bar{\partial}_f)}^*(D) = Ker(\bar{\partial}_f)/Im(\bar{\partial}_f) \cong \mathcal{H}^* \oplus \overline{Im(\bar{\partial}_f)}/Im(\bar{\partial}_f),$$

which is finite dimensional by Theorem 3.4. By Corollary IV.1.13 of [Go]: if a closed operator from a Banach space to another Banach space has finite cokernel, it must have closed range; we can conclude that $Im(\bar{\partial}_f)$ is closed. Now by Theorem 3.6, $Im(\bar{\partial}_f^*)$ is also closed. Thus we have

Theorem 3.7 (strong Hodge decomposition).

$$L_k^2(D) = \mathcal{H}^k \oplus Im(\bar{\partial}_f) \oplus Im(\bar{\partial}_f^*) \quad (3.5)$$

and the isomorphism

$$H_{((2), \bar{\partial}_f)}^k(D) \cong \mathcal{H}^k \quad (3.6)$$

For any

$$\psi \in (\mathcal{H}^*)^\perp = Ker(\bar{\partial}_f)^\perp + Ker(\bar{\partial}_f^*)^\perp,$$

let $\psi = \psi_1 + \psi_2 + \psi_3$ be the orthogonal decomposition of ψ into $Ker(\bar{\partial}_f)^\perp \cap Ker(\bar{\partial}_f^*)$, $Ker(\bar{\partial}_f)^\perp \cap Ker(\bar{\partial}_f^*)^\perp$ and $Ker(\bar{\partial}_f) \cap Ker(\bar{\partial}_f^*)^\perp$.

By Theorem 3.6 and closeness of range of $\bar{\partial}_f$ and $\bar{\partial}_f^*$, we then have

$$\begin{aligned} \langle \Delta_f \psi, \psi \rangle &= ||\bar{\partial}_f(\psi)||^2 + ||\bar{\partial}_f^*(\psi)||^2 \\ &= ||\bar{\partial}_f(\psi_1 + \psi_2)||^2 + ||\bar{\partial}_f^*(\psi_2 + \psi_3)||^2 \\ &\geq c||\psi_1 + \psi_2||^2 + c||\psi_2 + \psi_3||^2 \\ &= c(||\psi_1||^2 + 2||\psi_2||^2 + ||\psi_3||^2) \\ &\geq c||\psi||^2 \end{aligned}$$

for some positive constant c . That's, a spectral gap exists between 0 and other spectra of Δ_f . By Proposition 3.6, the range of Δ_f is closed. So we obtain

Corollary 3.8. *Let D be a bounded strongly pseudoconvex domain in \mathbb{C}^n with C^∞ smooth boundary. Let f be a holomorphic function in D without critical points on the boundary ∂D . Then the twisted Laplacian Δ_f has finite dimensional kernel and there exists a spectral gap between 0 and other spectra of Δ_f . The complex $(L^2(D), \bar{\partial}_f)$ has finite dimensional cohomology for $0 \leq p \leq n$ and zero cohomology for $n < p \leq 2n$.*

Now Theorem 3.7 and Corollary 3.8 gives Theorem 1.1.

Notation 3.9. Similar to the equality $I = \Delta N + P$ for the $\bar{\partial}$ -Neumann problem, let $P : L^2(D) \rightarrow \mathcal{H}^*(D)$ be the projection operator and $G : L^2(D) \rightarrow \text{Dom}(\Delta_f)$ be the Green operator, we have the decomposition

$$I = \Delta_f G + P.$$

4. GLOBAL REGULARITY

The operator Δ_f has the following expansion:

$$\Delta_f = \Delta_{\bar{\partial}} + \sum_{i,j} (f_{ij} (d\bar{z}_j \wedge)^* dz_i \wedge + \bar{f}_{ij} d\bar{z}_i \wedge (dz_j \wedge)^*) + \sum_{i=1}^n |f_i|^2.$$

Here $\Delta_{\bar{\partial}} = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$ and the last two summands, denoted by L_f and $|\nabla f|^2$ respectively, are of order 0. Hence Δ_f is an elliptic operator of second order and has the interior regularity estimate. In the following, we say that an operator T is globally regular if and only if it preserves $\mathcal{A}(\bar{D})$. T is exactly regular if it maps $W^s(D)$ continuously to itself for any non-negative integers. Exact regularity *a priori* means global regularity by Sobolev imbedding theorem.

To obtain a global estimate, we need sharper estimate about $\bar{\partial}$ -Neumann operator on strongly pseudoconvex domains. We state the results needed in our proof in the following.

Theorem 4.1. *For the $\bar{\partial}$ -Neumann problem on a strongly pseudoconvex domain D with smooth boundary, let P be the projection to space of harmonic forms and $N_{(p,q)}$ be the Neumann operator on (p,q) -forms. Then there exists positive constants c_s depends only on s such that the following global estimates hold:*

- (1) P is exactly regular, i.e. it maps $W^s(D)$ continuously to itself.
- (2) $\|N_{(0,0)}\psi\|_{s+\frac{1}{2}} \leq c_s \|\psi\|_s$.

- (3) $\|\bar{\partial}^* N_{(p,q)} \psi\|_{s+\frac{1}{2}} + \|\bar{\partial} N_{(p,q)} \psi\|_{s+\frac{1}{2}} \leq c_s \|\psi\|_s$ for $q \geq 1$.
 (4) $\|N_{(p,q)} \psi\|_{s+1} \leq c_s \|\psi\|_s$ for all $q \geq 1$.

Proof. All of them are classical results for $\bar{\partial}$ -Neumann problem. For (1), see comments behind Corollary 5.2.7 and Theorem 6.2.2 in [Sh]. (3) and (4) is exactly Theorem 5.3.10 in [Sh]. For (2), by Equality 5.3.34 in [Sh], (3) and (4)

$$\|N_{(0,0)} \psi\|_{s+\frac{1}{2}} \lesssim \|N_{(0,1)} \bar{\partial} \psi\|_s \lesssim \|\bar{\partial} \psi\|_{s-1} \lesssim \|\psi\|_s.$$

□

Lemma 4.2. *Assume D is strongly pseudoconvex with smooth ∂D and f has no critical points on ∂D . u is a function in $\text{Dom}(\Delta_f)$. If*

$$\Delta u + |\nabla f|^2 u = g$$

for some function $g \in W_0^s(D)$, then $u \in W_0^s(D)$.

Notice that the 0^{th} cohomology $H_{((2), \bar{\partial} f)}^0$ is obviously zero, so 0 is not in the spectrum of Δ_f . Therefore by Corollary 3.8, the spectrum of $\Delta_f = \Delta + |\nabla f|^2$ has a positive lower bound and $\Delta + |\nabla f|^2$ has a bounded inverse G_0 with $\|G_0 g\| \leq c \|g\|$.

Proof. By definition, $u \in L_0^2(D)$. Now assume $u \in W_0^k(D)$ for some $k \leq s - \frac{1}{2}$. Let $u = Pu + u^\perp$ be the decomposition of u into a holomorphic function and the orthogonal part. Then there is

$$\Delta u^\perp = g - |\nabla f|^2 u, \quad Pg = P(|\nabla f|^2 u).$$

By Theorem 4.1, we have

$$u^\perp = N_{(0,0)}(g - |\nabla f|^2 u) \in W_0^{k+\frac{1}{2}}(D).$$

On the other hand, we have

$$P(|\nabla f|^2 Pu) + P(|\nabla f|^2 u^\perp) = P(|\nabla f|^2 u) = Pg \in W_0^s(D).$$

Then it follows that

$$P(|\nabla f|^2 u^\perp), P(|\nabla f|^2 Pu) \in W_0^{k+\frac{1}{2}}(D).$$

Now we have

$$\begin{aligned}
|||\nabla f|^2 Pu|||_{k+\frac{1}{2}} &= ||P(|\nabla f|^2 Pu) + \bar{\partial}^* N_{(0,1)} \bar{\partial}(|\nabla f|^2 Pu)|||_{k+\frac{1}{2}} \\
&\leq ||P(|\nabla f|^2 Pu)|||_{k+\frac{1}{2}} + ||\bar{\partial}^* N_{(0,1)} \bar{\partial}(|\nabla f|^2 Pu)|||_{k+\frac{1}{2}} \\
&\lesssim ||P(|\nabla f|^2 Pu)|||_{k+\frac{1}{2}} + ||\bar{\partial}(|\nabla f|^2 Pu)|||_k \\
&= ||P(|\nabla f|^2 Pu)|||_{k+\frac{1}{2}} + ||Pu \bar{\partial} |\nabla f|^2|||_k \\
&\lesssim ||P(|\nabla f|^2 Pu)|||_{k+\frac{1}{2}} + ||Pu|||_k \\
&\lesssim ||P(|\nabla f|^2 Pu)|||_{k+\frac{1}{2}} + ||u|||_k,
\end{aligned}$$

which gives

$$|\nabla f|^2 Pu \in W_0^{k+\frac{1}{2}}(D).$$

Then there is

$$|\nabla f|^2 u = |\nabla f|^2 Pu + |\nabla f|^2 u^\perp \in W_0^{k+\frac{1}{2}}(D).$$

Since $\Delta + |\nabla f|^2$ is elliptic in the interior of D and $|\nabla f|^2$ is nonzero on the boundary ∂D , we can apply elliptic estimate in the interior and divide by $|\nabla f|^2$ near the boundary to conclude that $u \in W_0^{k+\frac{1}{2}}(D)$. Now by induction, $u \in W_0^s(D)$ holds. \square

Remark 4.3. We can NOT expect that u has higher regularity than g , which will lead to the compactness of G_0 by Rellich's lemma. For example, when $|\nabla f|^2$ happens to be a positive constant c , which is the case when $f = \sum_{i=1}^n z^i$, G_0 will have a non-zero eigenvalue $\frac{1}{c}$ and the infinite dimensional Bergman space as the corresponding eigenspace. Thus G_0 can not be compact and that's why we use an indirect way to prove the strong decomposition theorem.

Proposition 4.4. *Assume D is strongly pseudoconvex with smooth ∂D and f has no critical points on ∂D . Then the Green operator G is exactly regular and $\mathcal{H}^* \subset \mathcal{A}^*(\bar{D})$.*

Proof. Assume $\Delta_f \varphi = \psi$ and $\psi \in W_p^s(D)$.

For $0 \leq p \leq n$, every $\varphi \in L_p^2(D)$ can be decomposed by types as $\varphi = \varphi^{p,0} + \varphi'$. We have

$$\Delta \varphi' = \psi' - |\nabla f|^2 \varphi' - (L_f \varphi)' \in L_p^2(D)$$

According to Theorem 4.1, $\varphi' \in W_p^1(D)$. As L_f always transform (p, q) -forms into sum of $(p-1, q+1)$ -forms and $(p+1, q-1)$ -forms, $\varphi^{p,0}$ does not contribute to the $(p, 0)$ component of $L_f \varphi$. Therefore

$$\begin{aligned}
\Delta \varphi^{p,0} + |\nabla f|^2 \varphi^{p,0} &= \psi^{p,0} - (L_f \varphi)^{p,0} \\
&= \psi^{p,0} - (L_f \varphi')^{p,0} \in W_p^1(D)
\end{aligned}$$

Since the Neumann boundary condition for $(p, 0)$ -forms are the same as that for functions, so by Lemma 4.2 $\varphi^{p,0} \in W_p^1(D)$. Now by induction, exact regularity for G_p holds.

For $n < p \leq 2n$, as no $(k, 0)$ -forms are involved, by Theorem 4.1 (4), the proof follows as the standard bootstrap argument.

Finally, let $\psi = 0$, then Sobolev's imbedding theorem guarantees that forms in \mathcal{H}^* are all smooth up to the boundary. \square

5. DIMENSION COMPUTATION

In this section, we will compute the dimension of the L^2 cohomology group $H_{((2), \bar{\partial}_f)}^p$ for $0 \leq p \leq n$.

Proposition 5.1. *The complex $(\mathcal{A}^*(\bar{D}), \bar{\partial}_f)$ is quasi-isomorphic to the L^2 complex $(L^2(D), \bar{\partial}_f)$.*

Proof. For any $\phi \in \mathcal{A}^*(\bar{D})$, $\bar{\partial}_f \phi = 0$, we can solve the $\bar{\partial}_f$ -Neumann problem to obtain:

$$\phi = P\phi + \bar{\partial}_f \bar{\partial}_f^* G\phi.$$

By Proposition 4.4, $P\phi \in \mathcal{A}^*(\bar{D})$ and $\bar{\partial}_f^* G\phi \in \mathcal{A}^{*-1}(\bar{D})$. This gives an isomorphism

$$H^*(\mathcal{A}^*(\bar{D}), \bar{\partial}_f) \cong \mathcal{H}^* \cong H_{((2), \bar{\partial}_f)}^*(D).$$

\square

If we ignore the L^2 condition, we have the smooth complex $(\mathcal{A}^*(D), \bar{\partial}_f)$, whose cohomology can be computed by the spectral sequence as follows.

This complex can be viewed as a total complex of the double complex $(\mathcal{A}^{*,*}(D), \bar{\partial}, \partial f \wedge)$ with horizontal operator $\bar{\partial}$ and vertical operator $\partial f \wedge$. We consider the spectral sequence associated to the filtration

$$\mathcal{F}^k \mathcal{A}(D) = \bigoplus_{i \geq k} \mathcal{A}^{i,*}$$

for $k \in \mathbb{Z}$. Since D is pseudo-convex, it is stein, therefore the first page is concentrated at the first column with $(k, 0)$ term given by holomorphic k -forms. Since the f_i 's have only finite common zeros, they form a regular sequence on D . Therefore the cohomology of the holomorphic Koszul complex

$$0 \rightarrow \Omega^0(D) \xrightarrow{\partial f \wedge} \Omega^1(D) \xrightarrow{\partial f \wedge} \dots \xrightarrow{\partial f \wedge} \Omega^n(D) \xrightarrow{\partial f \wedge} 0$$

which is E_2 , is concentrated at the top term $\Omega^n(D)/\partial f \wedge \Omega^{n-1}(D) \cong \text{Jac}(f)$. Thus the spectral sequence degenerate at the E_2 -stage and the cohomology of the smooth $\bar{\partial}_f$ complex is concentrated at the middle

dimension and is isomorphic to $\text{Jac}(f)$. Hence we obtain the following result.

Proposition 5.2.

$$H_{\bar{\partial}_f}^k(D) = \begin{cases} \text{Jac}(f) & k = n \\ 0 & k \neq n. \end{cases} \quad (5.1)$$

To construct the cohomology of the complex $(\mathcal{A}_c^*(D), \bar{\partial}_f)$ consisting of the forms with compact support, we want to use a homotopy introduced in [LLS].

Let ρ be a smooth function with compact support in D such that it equals to 1 in a neighborhood of $\text{Crit}(f)$. Define the following operator

$$V_f = \sum_{i=1}^n \frac{\bar{f}_i}{|\nabla f|^2} (dz_i \wedge)^* : \mathcal{A}^{*,*}(D \setminus \text{Crit}(f)) \rightarrow \mathcal{A}^{*-1,*}(D \setminus \text{Crit}(f))$$

A direct calculation gives the following result.

Lemma 5.3.

$$[df \wedge, V_f] = 1 \quad (5.2)$$

and

$$[\bar{\partial}, [\bar{\partial}, V_f]] = [df \wedge, [\bar{\partial}, V_f]] = [V_f, [\bar{\partial}, V_f]] = 0 \quad (5.3)$$

Define two operators on D :

$$T_\rho = \rho + (\bar{\partial}\rho)V_f \frac{1}{1 + [\bar{\partial}, V_f]}, \quad R_\rho = (1 - \rho)V_f \frac{1}{1 + [\bar{\partial}, V_f]} \quad (5.4)$$

Lemma 5.4.

$$[\bar{\partial}_f, R_\rho] = 1 - T_\rho \quad \text{on} \quad \mathcal{A}^*(D) \quad (5.5)$$

Proof. By Lemma 5.3,

$$\begin{aligned} [\bar{\partial}_f, R_\rho] &= [\bar{\partial}_f, 1 - \rho]V_f \frac{1}{1 + [\bar{\partial}, V_f]} + (1 - \rho)[\bar{\partial}_f, V_f] \frac{1}{1 + [\bar{\partial}, V_f]} \\ &= (-\bar{\partial}\rho)V_f \frac{1}{1 + [\bar{\partial}, V_f]} + (1 - \rho)(1 + [\bar{\partial}, V_f]) \frac{1}{1 + [\bar{\partial}, V_f]} \\ &= 1 - T_\rho \quad \text{on} \quad \mathcal{A}^*(D) \end{aligned}$$

□

The Lemma 5.4 built a homotopy from $(\mathcal{A}(D), \bar{\partial}_f)$ to $(\mathcal{A}_c(D), \bar{\partial}_f)$, hence we have

Proposition 5.5.

$$H_{\bar{\partial}_f}^*(D) \cong H_{(c, \bar{\partial}_f)}^*(D). \quad (5.6)$$

Proof. Let $i : \mathcal{A}_c^k(D) \rightarrow \mathcal{A}^k(D)$ be the inclusion. Since $\bar{\partial}_f(\mathcal{A}_c(D)) \subset \bar{\partial}_f(\mathcal{A}(D))$, we have the well-defined homomorphism $i_* : H_{(c, \bar{\partial}_f)}^k(D) \rightarrow H_{\bar{\partial}_f}^k(D)$. Assume that $[i(b)] = 0 \in H_{\bar{\partial}_f}^k(D)$, then there exists a $c \in \mathcal{A}^{k-1}(D)$ such that $b = \bar{\partial}_f c$. By Lemma 5.4, we have

$$b = \bar{\partial}_f(T_\rho c + R_\rho \bar{\partial}_f c) = \bar{\partial}_f(T_\rho + R_\rho b),$$

where $T_\rho + R_\rho b \in \mathcal{A}_c(D)$. This shows that $[b]$ is the zero class in $H_{(c, \bar{\partial}_f)}^k(D)$. Hence i_* is injective. On the other hand, if $\bar{\partial}_f b = 0$ for $b \in \mathcal{A}^k(D)$, then $b = T_\rho b + \bar{\partial}_f R_\rho b$, which shows that i_* is also surjective. \square

Let us check R_ρ more carefully. In a small neighborhood of $\text{Crit}(f)$, $R_\rho = 0$. Outside such a neighborhood,

$$R_\rho = (1 - \rho)V_f \sum_{k=1}^n (-1)^k [\bar{\partial}, V_f]^k$$

Here V_f is of order 0 and

$$\begin{aligned} [\bar{\partial}, V_f] &= \sum_{i,j} \left[\frac{\partial}{\partial \bar{z}_i} d\bar{z}_i \wedge, \frac{\bar{f}_j}{|\nabla f|^2} (dz_j \wedge)^* \right] \\ &= \sum_{i,j} \frac{\partial}{\partial \bar{z}_i} \left(\frac{\bar{f}_j}{|\nabla f|^2} \right) d\bar{z}_i \wedge (dz_j \wedge)^* \end{aligned}$$

is also of order 0. So R_ρ is actually smooth and bounded, and it defines a bounded operator from $\mathcal{A}(\bar{D})$ to itself. Now using the homotopy in Lemma 5.4, we can also have the following result.

Proposition 5.6.

$$H^*(\mathcal{A}(\bar{D}), \bar{\partial}_f) \cong H_{\bar{\partial}_f}^*(D). \quad (5.7)$$

Combining the results of Proposition 5.1, 5.2, 5.5 and 5.6, we obtain Theorem 1.2.

Remark 5.7. When D is only pseudoconvex with smooth boundary, interior regularity of Δ_f shows $\mathcal{H}^* \subset \mathcal{A}^*(D)$ and G preserves $\mathcal{A}^*(D)$. Thus similar argument like Proposition 5.1 can be applied to show $H^*(L^2(D) \cap \mathcal{A}^*(D)) \cong \mathcal{H}^*$. Moreover, like Proposition 5.6, T_ρ and R_ρ can be used to give an isomorphism between $H^k(L^2(D) \cap \mathcal{A}^*(D))$ and $H_{(c, \bar{\partial}_f)}^k$. So Theorem 1.2 still holds.

Likewise, $H_{(0,\bar{\partial}_f)}^k \cong H_{(c,\bar{\partial}_f)}^k$ by using T_ρ and R_ρ . Thus by Proposition 3.2, $H_{\bar{\partial}_f}^n(\mathcal{C})$ and $H_{\bar{\partial}_f}^n(\bar{D})$ are isomorphic when D is strongly pseudoconvex. We can also obtain this result by proving the following duality theorem.

Theorem 5.8. *We have the isomorphism*

$$H_{\bar{\partial}_f}^k(\mathcal{C}) \cong (H_{\bar{\partial}_f}^{2n-k}(\bar{D}))^* \quad (5.8)$$

Proof. The idea is to construct a pairing

$$\varphi, \psi \mapsto \int_D \varphi \wedge \psi$$

for $\varphi \in H_{\bar{\partial}_f}^{2n-k}(\bar{D})$ and $\psi \in H_{\bar{\partial}_f}^k(\mathcal{C})$. Almost the same proof as the Proposition 5.1.5 in [FK], except the arise of $\bar{\partial}f \wedge$, which gives a minus sign here, shows that this is indeed a pairing. To show it is non-degenerate, we only need Theorem 3.7 to take the role of the condition Z(q) in Proposition 5.1.5 in [FK]. \square

REFERENCES

- [Ce] S. Cecotti, *N = 2 Landau-Ginzburg Vs. Calabi-Yau models: Non-perturbative aspects*, Int. J. Mod. Phys. A6(1991) 1749-.
- [CR] A. Chiodo and Y. Ruan, *Landau-Ginzburg/Calabi-Yau correspondence of quintic three-fold via symplectic transformations*, Invent. Math. 182 (2010), no. 1, 117–165.
- [CV] S. Cecotti and C. Vafa, *Topological anti-topological fusion*, Nucl. Phys. B367(1991) 359-461.
- [EP] J. Eschmeier and M. Putinar, *Spectral decompositions and analytic sheaves*, The Clarendon Press Oxford University Press, New York, 1996, Oxford Science Publications.
- [Fa] H. Fan, *Schrödinger equations, deformation theory and tt^* -geometry*, arXiv:1107.1290 [math-ph].
- [FJ] T. Jarvis and A. Francis, *A brief survey of FJRW theory*, arXiv:1503.01223 [math.AG].
- [FJR] H. Fan, T. Jarvis and Y. Ruan, *The Witten equation, mirror symmetry and quantum singularity theory*, Ann. of Math. Vol 108, No. 3, 1-106.
- [FK] G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Princeton University Press and University of Tokyo Press, Princeton, New Jersey, 1972.
- [BS] H. P. Boas and E. J. Straube, *Equivalence of regularity for the Bergman projection and the $\bar{\partial}$ -Neumann operator*, Manuscripta Math. 67:1 (1990), 25-33.
- [GMW] G. Gaiotto, G. Moore, and E. Witten, *Algebra of the infrared: string field theoretic structures in massive $N = (2, 2)$ field theory in two dimension*, in preparation.
- [Go] S. Goldberg, *Unbounded Linear Operators: Theory and Applications*, Dover, New York, 1985.

- [GT] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations*, Springer, 2001.
- [KKS] M. Kapranov, M. Kontsevich and Y. Soibelman, *Algebra of the infrared and secondary polytopes*, arXiv: 1408.2673v1[math.SG].
- [LLS] C.-C. Li, S. Li and K. Saito, *Primitive forms via polyvector fields*, arXiv:1311.1659 [math.AG].
- [Pu] M. Putinar, *Private communication*, May, 2015.
- [Sh] S.-C. Chen and M.-C. Shaw, *Partial differential equations in several complex variables*, AMS, Providence, RI, 2001.
- [ST] K. Saito, and A. Takahashi, *From primitive forms to Frobenius manifolds*, preprint, 2008.
- [Ta] J. L. Taylor, *A joint spectrum for several commuting operators*, J. Functional Analysis 6 (1970), 172

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